Exchange economy with perfect complements and max utility functions

In an economy there are two consumers, A and B, who are endowed with two commodities, x and y. The preferences of these consumers are given by the following utility functions:

$$u_A = \min\{x_A, y_A\}$$

$$u_B = \max\{x_B, y_B\}$$

Consumer A has an initial endowment $\omega_A = (10, 0)$, while consumer B has an initial endowment $\omega_B = (0, 10)$. Assuming that consumers take prices as given, answer the following:

What relative price equilibrates the market? What are the consumption bundles of both consumers after trade?

Solution

Let prices be (p_x, p_y), with $p_x > 0$ and $p_y > 0$, and define the relative price

$$r = \frac{p_x}{p_y}$$

Consumer A
Consumer A solves

$$\max_{x_A, y_A} \min\{x_A, y_A\}$$

subject to

$$p_x x_A + p_y y_A \leq 10 p_x$$

Since $u_A = \min\{x_A, y_A\}$, consumer A maximizes utility by choosing the two goods in equal quantities

$$x_A = y_A$$

Using the budget constraint,

$$p_x x_A + p_y x_A = 10 p_x$$

$$x_A(p_x + p_y) = 10 p_x$$

Thus, consumer A's demand is

$$x_A^* = y_A^* = \frac{10 p_x}{p_x + p_y}$$

or, in terms of r,

$$x_A^* = y_A^* = \frac{10r}{1+r}$$

Consumer B
Consumer B solves

$$\max_{x_B, y_B} \max\{x_B, y_B\}$$

subject to

$$p_x x_B + p_y y_B \leq 10 p_y$$

Since $u_B = \max\{x_B, y_B\}$, consumer B wants to make one of the two coordinates as large as possible. Therefore, B spends all income on the cheaper good

$$\text{if } p_x < p_y \quad \Rightarrow \quad (x_B^*, y_B^*) = \left(\frac{10 p_y}{p_x}, 0\right) = \left(\frac{10}{r}, 0\right)$$

$$\text{if } p_x > p_y \quad \Rightarrow \quad (x_B^*, y_B^*) = (0, 10)$$

If prices are equal, then consumer B is indifferent between buying only x or only y, so the demand correspondence is

$$\text{if } p_x = p_y \quad \Rightarrow \quad (x_B^*, y_B^*) \in \{(10, 0), (0, 10)\}$$

Market clearing
The total endowment in the economy is

$$\bar{x} = 10 \qquad \bar{y} = 10$$

We now check whether some relative price can clear both markets

Case 1: $r < 1$
Then $p_x < p_y$, so consumer B demands only good x

$$(x_B^*, y_B^*) = \left(\frac{10}{r}, 0\right)$$

Total demand for good y is then only

$$y_A^* + y_B^* = \frac{10r}{1+r}$$

Since $r < 1$,

$$\frac{10r}{1+r} < 10$$

so the market for good y cannot clear

Case 2: $r > 1$
Then $p_x > p_y$, so consumer B demands only good y

$$(x_B^*, y_B^*) = (0, 10)$$

Total demand for good x is then only

$$x_A^* + x_B^* = \frac{10r}{1+r}$$

Since $r > 1$,

$$\frac{10r}{1+r} < 10$$

so the market for good x cannot clear

Case 3: $r = 1$
Then consumer A demands

$$(x_A^*, y_A^*) = (5, 5)$$

while consumer B demands either

$$(10, 0) \qquad \text{or} \qquad (0, 10)$$

If consumer B chooses $(10, 0)$, total demand is

$$(15, 5)$$

If consumer B chooses $(0, 10)$, total demand is

$$(5, 15)$$

In neither case do both markets clear

Therefore, there is no positive relative price that clears both markets

There is no Walrasian equilibrium with positive prices in this economy

Hence, there is no equilibrium relative price and no equilibrium allocation after trade